

DYNKIN games in a general framework

Magdalena KOBYLANSKI ^{*} Marie-Claire QUENEZ [†]

Marc ROGER DE CAMPAGNOLLE [‡]

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Abstract

We revisit the Dynkin game problem in a general framework, improve classical results and relax some assumptions. The criterion is expressed in terms of families of random variables indexed by stopping times. We construct two nonnegative supermartingales families J and J' whose finitness is equivalent to the Mokobodski's condition. Under some weak right-regularity assumption, the game is shown to be fair and $J - J'$ is shown to be the common value function. Existence of saddle points is derived under some weak additional assumptions. All the results are written in terms of random variables and are proven by using only classical results of probability theory.

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Introduction

In this paper, the Dynkin game problem is revisited in the general framework of families of random variables indexed by stopping times. The criterion is given for each pair of stopping times (τ, σ) by

$$I_0(\tau, \sigma) := E[\xi(\tau)\mathbb{1}_{\{\tau \leq \sigma\}} + \zeta(\sigma)\mathbb{1}_{\{\sigma < \tau\}}].$$

Studying the Dynkin game problem consists in proving that, under suitable conditions, the game is fair that is,

$$\inf_{\sigma} \sup_{\tau} I_0(\tau, \sigma) = \sup_{\tau} \inf_{\sigma} I_0(\tau, \sigma),$$

in characterizing this common value function, and finally in proving the existence of saddle points.

Recall that this problem has been largely studied in the literature in the framework of processes. We mention, among others, Bismut (1979), Alario-Nazaret, Lepeltier and Marchal (1982). We stress on that this approach relies on sophisticated results of the General Theory of Processes and Optimal Stopping Theory.

^{*}(LAMA - UMR 8050) – Université Paris Est
magdalena.kobylanski@univ-mlv.fr

[†]Laboratoire de Probabilités et Modèles Aléatoires (L.P.M.A.) – Université Denis DIDEROT – Paris 7 / Inria
quenez@math.jussieu.fr

[‡]Laboratoire de Probabilités et Modèles Aléatoires (L.P.M.A.) – Université Denis DIDEROT – Paris 7
marc@math.jussieu.fr

Recently, Kobylanski and Quenez (2011) have revisited the optimal stopping problem in the case of a reward given by a *family of random variables* indexed by stopping times. This notion is very general and includes the case of processes as a particular case. This setup has appeared as relevant and appropriated as it allows, from the one hand, to release the hypotheses made on the reward, and on the second hand, to make simpler proofs using simpler tools.

In the present work, the setup of families of random variables indexed by stopping times allows to solve the Dynkin game problem under very weak assumptions by using only classical tools of Probability Theory.

The paper is organised as follows. In section 1, we introduce the Dynkin game problem. In section 2, we construct two $[0, +\infty]$ -valued supermartingale families J and J' . In the case of processes, this construction is classical (see for example Alario-Nazaret, Lepeltier and Marchal (1982)) and is done under the so called Mokobodski's condition on ξ and ζ . In the present work, we do not need any condition of this type in order to define J and J' . Mokobodski's condition then appears to be equivalent to the fact that $J(0)$ or $J'(0)$ is finite. In this case, $J(0) - J'(0)$ is well defined and naturally appears as the common value function candidate. In section 3, when $J(0) < +\infty$, under the assumption that the families ξ and $-\zeta$ are right-upper semicontinuous along stopping times in expectation, the game is shown to be fair and the common value function is proven to be equal to $J(0) - J'(0)$. In section 4, under some additional assumptions, we derive the existence of saddle points. At last, comes the Appendix where, in the first part, we briefly recall some results of Kobylanski and Quenez (2011) which are used in this paper and, in the second part, we provide two lemmas used to prove the existence result.

We point out that, whereas in the previous works, the proof of the existence of saddle points relied on some highly sophisticated tools of the General Theory of Processes, the one given in this paper does not require any prerequisite and is only done by using classical probability results.

We introduce some notation. Let $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right continuity and augmentation by the null sets $\mathcal{F} = \mathcal{F}_T$. We suppose that \mathcal{F}_0 contains only sets of probability 0 or 1. The time horizon is a fixed constant T in $]0, \infty[$. We denote by \mathcal{T} the collection of stopping times of \mathbb{F} with values in $[0, T]$. More generally, for any stopping times S , we denote by \mathcal{T}_S (resp. \mathcal{T}_{S+}) the class of stopping times $\theta \in \mathcal{T}$ with $\theta \geq S$ a.s. (resp. $\theta > S$ a.s. on $\{S < T\}$ and $\theta = T$ a.s. on $\{S = T\}$).

We also define $\mathcal{T}_{[S, S']}$ the set of $\theta \in \mathcal{T}$ with $S \leq \theta \leq S'$ a.s.

We use the following notation: for real valued random variables X and X_n , $n \in \mathbb{N}$, “ $X_n \uparrow X$ ” stands for “the sequence (X_n) is nondecreasing and converges to X a.s.”.

1 Dynkin games

In this section, we present the Dynkin game problem in the framework of families of random variables.

Definition 1.1 — A family of $\bar{\mathbb{R}}$ -valued random variables $\phi = (\phi(\theta), \theta \in \mathcal{T})$ is said to be *admissible* if it satisfies the following conditions

- 1) for all $\theta \in \mathcal{T}$ $\phi(\theta)$ is a \mathcal{F}_θ -measurable random variable (r.v.),
- 2) for all $\theta, \theta' \in \mathcal{T}$, $\phi(\theta) = \phi(\theta')$ a.s. on $\{\theta = \theta'\}$.

Remark 1.2 — Recall that the notion of admissible families is very general and includes the case of processes as a particular case. Indeed, if $(\phi_t)_{t \in \mathbb{R}_+}$ is a progressive process, the family of random variables $(\phi(\theta), \theta \in \mathcal{T})$ defined by $\phi(\theta) = \phi_\theta$ is admissible.

An admissible family $(\phi(\theta), \theta \in \mathcal{T})$ satisfying $E[\text{ess sup}_{\theta \in \mathcal{T}} \phi(\theta)^-] < +\infty$ is said to be a *supermartingale family* (resp. a *martingale family*) if for any $\theta, \theta' \in \mathcal{T}$ such that $\theta \geq \theta'$ a.s.,

$$E[\phi(\theta) | \mathcal{F}_{\theta'}] \leq \phi(\theta') \quad \text{a.s.}, \quad (\text{resp. } E[\phi(\theta) | \mathcal{F}_{\theta'}] = \phi(\theta') \quad \text{a.s.}).$$

Let $\phi = (\phi(\theta), \theta \in \mathcal{T})$ be an admissible family and let $\tau, \sigma \in \mathcal{T}$ such that $\tau \leq \sigma$ a.s. The family $(\phi(\theta), \theta \in \mathcal{T}_{[\tau, \sigma]})$ is said to be a martingale family if $(\phi((\theta \vee \tau) \wedge \sigma), \theta \in \mathcal{T})$ is a martingale family.

In order to simplify notation, we say that two admissible families $\phi = (\phi(\theta), \theta \in \mathcal{T})$ and $\phi' = (\phi'(\theta), \theta \in \mathcal{T})$ satisfy $\phi \geq \phi'$ if, for each $\theta \in \mathcal{T}$, $\phi(\theta) \geq \phi'(\theta)$ a.s. Similarly, we define the relations \leq and $=$.

Concerning the optimal stopping problem in this general framework, we refer the reader to Kobylanski and Quenez (2011). The main results are recalled in the Appendix.

We now introduce the Dynkin game problem.

Let $\xi = (\xi(\theta), \theta \in \mathcal{T})$ and $\zeta = (\zeta(\theta), \theta \in \mathcal{T})$ be two admissible families satisfying:

$$\mathbb{E} \left[\text{ess sup}_{\theta \in \mathcal{T}} (\xi(\theta))^- \right] < +\infty \quad \text{and} \quad \mathbb{E} \left[\text{ess sup}_{\theta \in \mathcal{T}} (\zeta(\theta))^+ \right] < +\infty, \quad (1)$$

and we suppose that $\xi(T) = \zeta(T) = 0$ a.s. This condition is not really a restriction (see below Remark 1.5).

We consider the following game with two players. The rule of the game is as follows. Each of the players has to choose a stopping time, denoted by τ for the first player and σ for the second one. The game stops at $\tau \wedge \sigma$. On $\{\tau \leq \sigma\}$, the second player pays the amount $\xi(\tau)$ to the first one and on $\{\sigma < \tau\}$, the first player pays the amount $-\zeta(\sigma)$ to the second one. In other words, at time $\tau \wedge \sigma$, the first player receives the payoff $\xi(\tau)\mathbb{1}_{\{\tau \leq \sigma\}} + \zeta(\sigma)\mathbb{1}_{\{\sigma < \tau\}}$ and the second one receives the opposite amount.

At time 0, the main of the first (resp. second player) is to maximize the expectation of his payoff. More precisely, if the second player has chosen to stop at time σ , the first one wants to maximize over τ the quantity $I_0(\tau, \sigma) := E[\xi(\tau)\mathbb{1}_{\{\tau \leq \sigma\}} + \zeta(\sigma)\mathbb{1}_{\{\sigma < \tau\}}]$. If the first player has chosen to stop at time τ , the second one wants to minimize over σ the quantity $I_0(\tau, \sigma)$. Now, the players are unwilling to risk. Thus, the first one wants to find a strategy τ which maximizes the quantity $\inf_{\sigma} I_0(\tau, \sigma)$ and the second one wants to find a strategy σ which minimizes the quantity $\sup_{\tau} I_0(\tau, \sigma)$.

By making the problem dynamic, we thus are led to introduce for each $\theta \in \mathcal{T}$, for each $(\tau, \sigma) \in \mathcal{T}_\theta^2$, the *criterion at time θ* is defined by:

$$I_\theta(\tau, \sigma) := \mathbb{E} \left[\xi(\tau)\mathbb{1}_{\{\tau \leq \sigma\}} + \zeta(\sigma)\mathbb{1}_{\{\sigma < \tau\}} \mid \mathcal{F}_\theta \right]. \quad (2)$$

Also, the *first* value function at time θ is given by

$$\underline{V}(\theta) := \text{ess sup}_{\tau \in \mathcal{T}_\theta} \text{ess inf}_{\sigma \in \mathcal{T}_\theta} I_\theta(\tau, \sigma), \quad (3)$$

which corresponds to the best gain the first player can expect when faced to a wise partner.

Also, the *second* value function at time θ is given by

$$\bar{V}(\theta) := \operatorname{ess\,inf}_{\sigma \in \mathcal{T}_\theta} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_\theta} I_\theta(\tau, \sigma), \quad (4)$$

which corresponds to the least lost the second player can expect.

For each $\theta \in \mathcal{T}$, the inequality $V(\theta) \leq \bar{V}(\theta)$ a.s. clearly holds.

The game is considered to be *fair* if $V(\theta) = \bar{V}(\theta)$ a.s. and this quantity is then referred as the *common* value function.

We now introduce the following definition.

Definition 1.3 — Let $\theta \in \mathcal{T}$. A pair $(\hat{\tau}, \hat{\sigma}) \in \mathcal{T}_\theta^2$ is called a θ -*saddle point* if, for each $(\tau, \sigma) \in \mathcal{T}_\theta^2$:

$$I_\theta(\tau, \hat{\sigma}) \leq I_\theta(\hat{\tau}, \hat{\sigma}) \leq I_\theta(\hat{\tau}, \sigma) \quad \text{a.s.} \quad (5)$$

In the study of the Dynkin game problem, the first object is to provide some sufficient conditions under which the game is fair and, in this case, the characterization of the common value function. The second one is to address the question of the existence of saddle points.

Remark 1.4 — By classical results on game problems, for each $\theta \in \mathcal{T}$, a pair $(\hat{\tau}, \hat{\sigma})$ is a θ -saddle point if and only if $V(\theta) = \bar{V}(\theta)$ a.s. and the essential infimum in (4) and the essential supremum in (3) are respectively attained at $\hat{\sigma}$ and $\hat{\tau}$. Hence, if, at initial time θ , $(\hat{\tau}, \hat{\sigma})$ is a θ -saddle point, then $\hat{\tau}$ is an optimal strategy for the first player and $\hat{\sigma}$ is an optimal strategy for the second one.

Remark 1.5 — First, one can easily prove that the assumption $\xi(T) = \zeta(T)$ a.s. is not restrictive because the criterion does not depend on the terminal cost $\zeta(T)$.

Second, the additional assumption that this common value is equal to 0 is no more restrictive. Indeed, let ξ and ζ be two admissible families such that $\xi(T) = \zeta(T)$ a.s. (but not necessarily equal to 0). Let us prove that the associated Dynkin game problem reduces to a Dynkin game with terminal rewards equal to 0. Let us make the following change of variables:

$$\forall \theta \in \mathcal{T}, \quad \xi'(\theta) := \xi(\theta) - \mathbb{E}[\xi(T) | \mathcal{F}_\theta] \quad \text{and} \quad \zeta'(\theta) := \zeta(\theta) - \mathbb{E}[\xi(T) | \mathcal{F}_\theta],$$

(by supposing that the random variable $\xi(T)$ is integrable). We clearly have $\xi'(T) = \zeta'(T) = 0$ a.s. and, for every $\theta \in \mathcal{T}$ and every $(\tau, \sigma) \in \mathcal{T}_\theta^2$, the associated criterion $I_\theta(\tau, \sigma)$ can be written:

$$I_\theta(\tau, \sigma) = \mathbb{E} \left[\xi'(\tau) \mathbb{1}_{\{\tau \leq \sigma\}} + \zeta'(\sigma) \mathbb{1}_{\{\sigma < \tau\}} | \mathcal{F}_\theta \right] + \mathbb{E}[\xi(T) | \mathcal{F}_\theta].$$

The game problem thus reduces to the one associated to the costs ξ' and ζ' . Also, if we suppose that

$$\mathbb{E} \left[\operatorname{ess\,sup}_{\theta \in \mathcal{T}} \left(\xi(\theta) - \mathbb{E}[\xi(T) | \mathcal{F}_\theta] \right)^- \right] < +\infty \quad \text{and} \quad \mathbb{E} \left[\operatorname{ess\,sup}_{\theta \in \mathcal{T}} \left(\zeta(\theta) - \mathbb{E}[\xi(T) | \mathcal{F}_\theta] \right)^+ \right] < +\infty,$$

then, the new rewards ξ' and ζ' clearly satisfy the integrability condition (1).

2 Preliminary results

In this section, we first provide the construction of two $[0, +\infty]$ -valued supermartingale families J and J' . In the case of processes, this construction is classical (see for example Alario-Nazaret, Lepeltier and Marchal (1982)) and is done under the so called Mokobodski's condition on ξ and ζ . We stress on that in the present work, we do not need any condition of this type in order to define J and J' . Mokobodski's condition then appears to be equivalent to the fact that $J(0)$ or $J'(0)$ is finite. Under this condition, $J - J'$ is well defined and naturally appears as the common value function candidate.

2.1 Construction of J and J'

For each $\theta \in \mathcal{T}$, set

$$J_0(\theta) := 0 \quad \text{and} \quad J'_0(\theta) := 0$$

and, let us introduce for each $n \in \mathbb{N}$,

$$J_{n+1}(\theta) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_\theta} \mathbb{E} [J'_n(\tau) + \xi(\tau) \mid \mathcal{F}_\theta], \quad (6)$$

$$J'_{n+1}(\theta) := \operatorname{ess\,sup}_{\sigma \in \mathcal{T}_\theta} \mathbb{E} [J_n(\sigma) - \zeta(\sigma) \mid \mathcal{F}_\theta], \quad (7)$$

which are well defined by the following lemma.

Lemma 2.1 — *For each $n \in \mathbb{N}$ and each $\theta \in \mathcal{T}$, the random variables $J_n(\theta)$ and $J'_n(\theta)$ are well defined and nonnegative, namely $[0, +\infty]$ -valued. Moreover, the families $J_n = (J_n(\theta), \theta \in \mathcal{T})$ and $J'_n = (J'_n(\theta), \theta \in \mathcal{T})$ are admissible and satisfy*

$$J_{n+1} = \mathcal{R}(J'_n + \xi) \quad \text{and} \quad J'_{n+1} = \mathcal{R}(J_n - \zeta),$$

where \mathcal{R} is the Snell envelope operator. In other words, the families J_{n+1} and J'_{n+1} are the smallest supermartingale families greater (almost surely) than $J'_n + \xi$ and respectively $J_n - \zeta$.

Proof — Let us show this property by induction. First, it clearly holds for J_0 and J'_0 since they are equal to 0. Let us suppose that for a fixed $n \in \mathbb{N}$, for each $\theta \in \mathcal{T}$, $J_n(\theta)$ and $J'_n(\theta)$ are well-defined $\bar{\mathbb{R}}^+$ -valued random variables. We then show that this property still holds for $n+1$. Let $\theta \in \mathcal{T}$. Since, for every $\tau \in \mathcal{T}_\theta$, $J'_n(\tau) \geq 0$ a.s. and, since by assumption (1),

$$\mathbb{E} \left[\operatorname{ess\,sup}_{\theta \in \mathcal{T}} (\xi(\theta))^- \right] < +\infty,$$

the random variable $J_{n+1}(\theta)$ is well defined by (6). Using (6), the induction hypothesis (non-negativity of J'_n) and the equality $\xi(T) = 0$ a.s., we derive that

$$J_{n+1}(\theta) \geq \mathbb{E} [J'_n(T) + \xi(T) \mid \mathcal{F}_\theta] \geq 0 \quad \text{a.s.}$$

By similar arguments and since by assumption (1),

$$E \left[\operatorname{ess\,sup}_{\theta \in \mathcal{T}} (\zeta(\theta))^+ \right] < +\infty,$$

we also have that $J'_{n+1}(\theta)$ is well defined and nonnegative. Moreover, the admissibility property of the families J_n and J'_n can be proven by induction. By classical results (see the Appendix), for each n , the families J_{n+1} and J'_{n+1} are the smallest supermartingale families respectively greater (almost surely) than $J'_n + \xi$ and $J_n - \zeta$. The proof is thus complete. \square

Lemma 2.2 — *The sequences of families $(J_n)_{n \in \mathbb{N}}$ and $(J'_n)_{n \in \mathbb{N}}$ are (almost surely) non-decreasing.*

Proof — The property can be proven by induction. By the previous lemma, we have $J_1 \geq 0 = J_0$ and $J'_1 \geq 0 = J'_0$. Let us now suppose that, for a fixed $n \in \mathbb{N}^*$, we have $J_n \geq J_{n-1}$ and $J'_n \geq J'_{n-1}$. We then have

$$\mathcal{R}(J'_n + \xi) \geq \mathcal{R}(J'_{n-1} + \xi) \quad \text{and} \quad \mathcal{R}(J_n - \zeta) \geq \mathcal{R}(J_{n-1} - \zeta),$$

which leads to $J_{n+1} \geq J_n$ and $J'_{n+1} \geq J'_n$. This concludes the proof. \square

For each $\theta \in \mathcal{T}$, let us define $J(\theta) := \limsup_{n \rightarrow +\infty} J_n(\theta)$ and $J'(\theta) = \limsup_{n \rightarrow +\infty} J'_n(\theta)$. We clearly have:

$$J(\theta) = \lim_{n \rightarrow +\infty} \uparrow J_n(\theta) \quad \text{a.s.} \quad \text{and} \quad J'(\theta) = \lim_{n \rightarrow +\infty} \uparrow J'_n(\theta) \quad \text{a.s.}$$

Note that the families $J = (J(\theta), \theta \in \mathcal{T})$ and $J' = (J'(\theta), \theta \in \mathcal{T})$ are clearly admissible and nonnegative (since, for each $n \in \mathbb{N}$, J_n and J'_n are themselves nonnegative).

Theorem 2.3 — *Let ξ and ζ be two admissible families satisfying integrability condition (1) with $\xi(T) = \zeta(T) = 0$ a.s. The associated families J and J' are then nonnegative supermartingale families which satisfy $J = \mathcal{R}(J' + \xi)$ and $J' = \mathcal{R}(J - \zeta)$ that is, for each $\theta \in \mathcal{T}$,*

$$J(\theta) = \text{ess sup}_{\tau \in \mathcal{T}_\theta} \mathbb{E}[J'(\tau) + \xi(\tau) | \mathcal{F}_\theta] \quad \text{a.s.}, \quad (8)$$

$$J'(\theta) = \text{ess sup}_{\sigma \in \mathcal{T}_\theta} \mathbb{E}[J(\sigma) - \zeta(\sigma) | \mathcal{F}_\theta] \quad \text{a.s.} \quad (9)$$

Moreover, J and J' are minimal in the following sense: if H and H' are two nonnegative supermartingale families such that $H \geq H' + \xi$ and $H' \geq H - \zeta$, then we have $J \leq H$ and $J' \leq H'$.

Remark 2.4 — From the second assertion of the above theorem, it clearly follows that J and J' are the minimal nonnegative supermartingale families which are solutions of the system (8) and (9). More precisely, if \bar{J} and \bar{J}' are two nonnegative supermartingale families satisfying (8) and (9) that is, $\bar{J} = \mathcal{R}(\bar{J}' + \xi)$ and $\bar{J}' = \mathcal{R}(\bar{J} - \zeta)$, then we have $J \leq \bar{J}$ and $J' \leq \bar{J}'$.

Before giving the proof of this above theorem, we first show that the limit of a non decreasing sequence of nonnegative supermartingale families is a supermartingale family. More precisely, the following lemma holds.

Lemma 2.5 — *For each $n \in \mathbb{N}$, let $\phi_n := (\phi_n(\theta), \theta \in \mathcal{T})$ be a nonnegative supermartingale family. Suppose that the sequence $(\phi_n)_{n \in \mathbb{N}}$ is non decreasing that is, for each $n \in \mathbb{N}$ and each $\theta \in \mathcal{T}$, $\phi_n(\theta) \leq \phi_{n+1}(\theta)$ a.s. The family $\phi = (\phi(\theta), \theta \in \mathcal{T})$ defined for each $\theta \in \mathcal{T}$ by $\phi(\theta) := \limsup_{n \rightarrow \infty} \phi_n(\theta)$ is then a nonnegative supermartingale family.*

Proof — One can easily prove that ϕ is an admissible family. Let us prove that it is a supermartingale family. Let $\theta, \theta' \in \mathcal{T}$ be such that $\theta \leq \theta'$ a.s. By the monotone convergence theorem for the conditional expectation, we get

$$E[\phi(\theta') | \mathcal{F}_\theta] = \lim_{n \rightarrow \infty} \uparrow E[\phi_n(\theta') | \mathcal{F}_\theta] \leq \lim_{n \rightarrow \infty} \uparrow \phi_n(\theta) = \phi(\theta) \quad \text{a.s.},$$

where the inequality follows from the supermartingale family property of ϕ_n , for each n . \square

Proof of Theorem 2.3 — From Lemma 2.5, since J and J' are the limit of non-decreasing sequences of nonnegative supermartingale families, they are nonnegative supermartingale families. For each $n \in \mathbb{N}$, we have:

$$J_{n+1} = \mathcal{R}(J'_n + \xi) \leq \mathcal{R}(J' + \xi).$$

By letting n tend to $+\infty$, we get that

$$J \leq \mathcal{R}(J' + \xi). \quad (10)$$

Now, for each $n \in \mathbb{N}$, $J_{n+1} \geq J'_n + \xi$. By letting n tend to $+\infty$, we derive that $J \geq J' + \xi$. By the supermartingale property of J and the characterization of $\mathcal{R}(J' + \xi)$ as the smallest supermartingale greater than $J' + \xi$, it follows that $J \geq \mathcal{R}(J' + \xi)$. This with (10) yields that $J = \mathcal{R}(J' + \xi)$. By similar arguments, one can easily derive that $J' = \mathcal{R}(J - \zeta)$.

It remains to show the second assertion. Let $H = (H(\theta))_{\theta \in \mathcal{T}}$ and $H' = (H'(\theta))_{\theta \in \mathcal{T}}$ be two nonnegative supermartingale families such that $H \geq H' + \xi$ and $H' \geq H - \zeta$. Now, the following lemma holds.

Lemma 2.6 — *For each $n \in \mathbb{N}$, $J_n \leq H$ and $J'_n \leq H'$.*

By letting n tend to $+\infty$ in the previous lemma, we get that $J \leq H$ and $J' \leq H'$, which ends the proof of Theorem 2.3. \square

Proof of Lemma 2.6 — Let us show the result by induction. It clearly holds for J_0 and J'_0 . Let us suppose that, for some fixed $n \in \mathbb{N}$, $J_n \leq H$ and $J'_n \leq H'$.

Let $\theta \in \mathcal{T}$. By using the induction hypothesis, the inequality $H \geq H' + \xi$ and the supermartingale property of H , we have, for each $\tau \in \mathcal{T}_\theta$,

$$\mathbb{E}[J'_n(\tau) + \xi(\tau) | \mathcal{F}_\theta] \leq \mathbb{E}[H'(\tau) + \xi(\tau) | \mathcal{F}_\theta] \leq \mathbb{E}[H(\tau) | \mathcal{F}_\theta] \leq H(\theta) \quad \text{a.s.}$$

By taking the essential supremum over $\tau \in \mathcal{T}_\theta$, we derive that $J_{n+1}(\theta) \leq H(\theta)$ a.s.

By similar arguments, we also get that $J'_{n+1}(\theta) \leq H'(\theta)$ a.s., which completes the proof of Lemma 2.6. \square

Note now that since $J \geq J' + \xi$ and $J' \geq J - \zeta$, it follows that $J(0) < +\infty$ if and only if $J'(0) < +\infty$.

2.2 Mokobodski's condition

Suppose now $J(0) < +\infty$. We can then define the admissible family Y by

$$Y := J - J',$$

and we have

$$\xi \leq Y \leq \zeta. \tag{11}$$

In particular, if $J(0) < +\infty$, then for each $\theta \in \mathcal{T}$, $\xi(\theta) \leq \zeta(\theta)$ a.s. In other words, if there exists $\nu \in \mathcal{T}$ such that $P(\xi(\nu) > \zeta(\nu)) > 0$, then $J(0) = J'(0) = +\infty$.

Moreover, by the second assertion of the previous theorem, the property $J(0) < +\infty$ is equivalent to the existence of two nonnegative supermartingale families H and H' such that $H \geq H' + \xi$ and $H' \geq H - \zeta$ with $H(0) < +\infty$ (or equivalently with $H'(0) < +\infty$). In other words, we have

Proposition 2.7 — *The following assertions are equivalent:*

- $J(0) < +\infty$;
- $J'(0) < +\infty$;
- Mokobodski's condition holds that is, there exist two nonnegative supermartingale families H and H' with $H(0) < +\infty$ (or equivalently with $H'(0) < +\infty$) such that

$$\xi \leq H - H' \leq \zeta.$$

Remark 2.8 — If for example, $\xi \leq 0 \leq \zeta$, Mokobodski's condition is then satisfied with $H = H' = 0$ ($= J = J'$) and we clearly have $\underline{V} = \bar{V} = 0$.

2.3 A sufficient condition to solve the Dynkin game problem

Under the above condition, $J - J'$ is well defined and naturally appears as the common value function candidate.

We have the following property.

Proposition 2.9 — *Suppose that $J(0) < +\infty$. Let $\theta \in \mathcal{T}$ and let $(\hat{\tau}, \hat{\sigma}) \in \mathcal{T}_\theta^2$ be such that $\hat{\tau}$ is optimal for $J(\theta)$, that is*

$$J(\theta) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_\theta} \mathbb{E} [J'(\tau) + \xi(\tau) | \mathcal{F}_\theta] = \mathbb{E} [J'(\hat{\tau}) + \xi(\hat{\tau}) | \mathcal{F}_\theta] \quad \text{a.s.},$$

and $\hat{\sigma}$ is optimal for $J'(\theta)$, that is

$$J'(\theta) = \operatorname{ess\,sup}_{\sigma \in \mathcal{T}_\theta} \mathbb{E} [J(\sigma) - \zeta(\sigma) | \mathcal{F}_\theta] = \mathbb{E} [J(\hat{\sigma}) - \zeta(\hat{\sigma}) | \mathcal{F}_\theta] \quad \text{a.s.}$$

Then, the game is fair, the common value function is equal to $Y(\theta) (= J(\theta) - J'(\theta))$ and $(\hat{\tau}, \hat{\sigma})$ is a θ -saddle point. We thus have

$$Y(\theta) = \underline{V}(\theta) = \bar{V}(\theta) = I_\theta(\hat{\tau}, \hat{\sigma}) \quad \text{a.s.} \quad (12)$$

Remark 2.10 — Note that if ξ is a supermartingale family, then for each $\theta \in \mathcal{T}$, (θ, T) is a θ -saddle point. Indeed, for each $(\tau, \sigma) \in \mathcal{T}_\theta^2$,

$$I_\theta(\theta, T) = I_\theta(\theta, \sigma) = E[\xi(\theta) | \mathcal{F}_\theta] = \xi(\theta) \quad \text{a.s.},$$

and $I_\theta(\tau, T) = E[\xi(\tau) | \mathcal{F}_\theta] \leq \xi(\theta) \quad \text{a.s.}$ and the common value function is equal to ξ (without any condition on ζ). Hence, if additionally, there exists $\nu \in \mathcal{T}$ such that $P(\xi(\nu) > \zeta(\nu)) > 0$, then $J(0) = +\infty$ (by (11)) even if for each θ , a θ -saddle point does exist.

This example shows that condition $J(0) < +\infty$, or equivalently Mokobodski's condition, is not necessary to have the existence of saddle points. We even have that the weaker condition $\xi \leq \zeta$ is not a necessary condition.

Proof — Let $\theta \in \mathcal{T}$. By the optimality criterion (see (22) in the Appendix), $(J(\tau), \tau \in \mathcal{T}_{[\theta, \hat{\tau}]})$ is a martingale family and $J(\hat{\tau}) = J'(\hat{\tau}) + \xi(\hat{\tau})$ a.s., that is $Y(\hat{\tau}) = \xi(\hat{\tau})$ a.s. Also, $(J'(\sigma), \sigma \in \mathcal{T}_{[\theta, \hat{\sigma}]})$ is a martingale family and $J'(\hat{\sigma}) = J(\hat{\sigma}) - \zeta(\hat{\sigma})$ a.s. that is $Y(\hat{\sigma}) = \zeta(\hat{\sigma})$ a.s. Since $Y = J - J'$, it follows that $(Y(\alpha), \alpha \in \mathcal{T}_{[\theta, \hat{\tau} \wedge \hat{\sigma}]})$ is a martingale family and hence that

$$\begin{aligned} Y(\theta) &= \mathbb{E} [Y(\hat{\tau} \wedge \hat{\sigma}) | \mathcal{F}_\theta] = \mathbb{E} [Y(\hat{\tau}) \mathbb{1}_{\{\hat{\tau} \leq \hat{\sigma}\}} + Y(\hat{\sigma}) \mathbb{1}_{\{\hat{\sigma} < \hat{\tau}\}} | \mathcal{F}_\theta] \\ &= \mathbb{E} [\xi(\hat{\tau}) \mathbb{1}_{\{\hat{\tau} \leq \hat{\sigma}\}} + \zeta(\hat{\sigma}) \mathbb{1}_{\{\hat{\sigma} < \hat{\tau}\}} | \mathcal{F}_\theta] = I_\theta(\hat{\tau}, \hat{\sigma}) \quad \text{a.s.} \end{aligned}$$

Let us now show that, for each $\sigma \in \mathcal{T}_\theta$, $Y(\theta) \leq I_\theta(\hat{\tau}, \sigma)$ a.s. Let $\sigma \in \mathcal{T}_\theta$. Since $(J(\tau), \tau \in \mathcal{T}_{[\theta, \hat{\tau}]})$ is a martingale family and J' is a supermartingale family, it follows that $Y = J - J'$ is a submartingale family on $[\theta, \hat{\tau} \wedge \sigma]$, which ensures that:

$$Y(\theta) \leq \mathbb{E} [Y(\hat{\tau} \wedge \sigma) | \mathcal{F}_\theta] \leq \mathbb{E} [\xi(\hat{\tau}) \mathbb{1}_{\{\hat{\tau} \leq \sigma\}} + \zeta(\sigma) \mathbb{1}_{\{\sigma < \hat{\tau}\}} | \mathcal{F}_\theta] = I_\theta(\hat{\tau}, \sigma) \quad \text{a.s.},$$

where the second inequality follows from the fact that $Y(\hat{\tau}) = \xi(\hat{\tau})$ a.s. and $Y(\sigma) \leq \zeta(\sigma)$ a.s. By similar arguments, one can show that, for each $\tau \in \mathcal{T}_\theta$, $I_\theta(\tau, \hat{\sigma}) \leq Y(\theta)$ a.s. We have thus proved that $(\hat{\tau}, \hat{\sigma})$ is a θ -saddle point for J and that equalities (12) hold. \square

Remark 2.11 — Let \bar{J} and \bar{J}' be two nonnegative supermartingale families satisfying integrability conditions (1) and such that $\bar{J} = \mathcal{R}(\bar{J}' + \xi)$ and $\bar{J}' = \mathcal{R}(\bar{J} - \zeta)$. The same proof shows that the above property still holds for \bar{J} and \bar{J}' .

More precisely, if $\bar{J}(0) < +\infty$ and if, for some $\theta \in \mathcal{T}$ and $(\hat{\tau}, \hat{\sigma}) \in \mathcal{T}_\theta^2$, $\hat{\tau}$ is optimal for $\bar{J}(\theta)$ and $\hat{\sigma}$ is optimal for $\bar{J}'(\theta)$, then $(\hat{\tau}, \hat{\sigma})$ is a θ -saddle point and we have:

$$\bar{J}(\theta) - \bar{J}'(\theta) = Y(\theta) = \underline{V}(\theta) = \bar{V}(\theta) \quad \text{a.s.}$$

In particular, we have $\bar{J}(\theta) - \bar{J}'(\theta) = J(\theta) - J'(\theta)$ a.s.

3 Existence and characterization of the common value function

Let us now introduce the following definition.

Definition 3.1 — An admissible family $(\phi(\theta), \theta \in \mathcal{T})$, satisfying $E[\text{ess sup}_{\theta \in \mathcal{T}} \phi(\theta)^-] < +\infty$ is said to be *right-(resp. left-) upper semicontinuous in expectation along stopping times (right-(resp. left-) USCE)* if for all $\theta \in \mathcal{T}$ and for all sequences of stopping times (θ_n) such that $\theta^n \downarrow \theta$ (resp. $\theta^n \uparrow \theta$)

$$E[\phi(\theta)] \geq \limsup_{n \rightarrow \infty} E[\phi(\theta_n)].$$

Theorem 3.2 — Suppose that $J(0) < +\infty$ and that the families $(\xi(\theta), \theta \in \mathcal{T})$ and $(-\zeta(\theta), \theta \in \mathcal{T})$ are right-USCE. Then, the game is fair and the common value function is equal to $Y (= J - J')$ that is, for each $\theta \in \mathcal{T}$,

$$Y(\theta) = \underline{V}(\theta) = \bar{V}(\theta) \quad \text{a.s.} \quad (13)$$

Proof — Let $\theta \in \mathcal{T}$. For each $\lambda \in]0, 1[$, we introduce

$$\tau^\lambda(\theta) := \text{ess inf} \{ \tau \in \mathcal{T}_\theta, \lambda J(\tau) \leq J'(\tau) + \xi(\tau) \text{ a.s.} \}.$$

and

$$\sigma^\lambda(\theta) := \text{ess inf} \{ \sigma \in \mathcal{T}_\theta, \lambda J'(\sigma) \leq J(\sigma) - \zeta(\sigma) \text{ a.s.} \}.$$

Recall that (see (25) in the Appendix), $\tau^\lambda(\theta)$ (resp. $\sigma^\lambda(\theta)$) is $(1 - \lambda)$ -optimal for $J(\theta)$ (resp. $J'(\theta)$). In order to simplify notation, in the sequel, $\tau^\lambda(\theta)$ (resp. $\sigma^\lambda(\theta)$) will be denoted by τ^λ (resp. σ^λ). Now, the following lemma holds.

Lemma 3.3 — For each $\lambda \in]0, 1[$ and each $(\sigma, \tau) \in \mathcal{T}_\theta^2$, we have

$$I_\theta(\tau, \sigma^\lambda) - (1 - \lambda)J'(\theta) \leq Y(\theta) \leq I_\theta(\tau^\lambda, \sigma) + (1 - \lambda)J(\theta) \quad \text{a.s.}$$

We postpone for a while the proof of this lemma and complete the proof Theorem 3.2. We clearly have that $\underline{V}(\theta) \leq \bar{V}(\theta)$ a.s. Hence, it is sufficient to prove that

$$\bar{V}(\theta) \leq Y(\theta) \leq \underline{V}(\theta) \quad \text{a.s.} \quad (14)$$

Now, the previous lemma yields that for each $\lambda \in]0, 1[$,

$$\text{ess sup}_{\tau \in \mathcal{T}_\theta} I_\theta(\tau, \sigma^\lambda) - (1 - \lambda)J'(\theta) \leq Y(\theta) \leq \text{ess inf}_{\sigma \in \mathcal{T}_\theta} I_\theta(\tau^\lambda, \sigma) + (1 - \lambda)J(\theta) \quad \text{a.s.},$$

which implies that

$$\bar{V}(\theta) - (1 - \lambda)J'(\theta) \leq Y(\theta) \leq \underline{V}(\theta) + (1 - \lambda)J(\theta) \quad \text{a.s.}$$

By letting λ tend to 1, we get inequalities (14). It follows that $\bar{V}(\theta) = Y(\theta) = \underline{V}(\theta)$ a.s., and completes the proof of Theorem 3.2. \square

It remains to prove Lemma 3.3 which actually shows that $(\sigma^\lambda, \tau^\lambda)$ is an $(1 - \lambda)$ -saddle point. *Proof of Lemma 3.3* — First, one can easily show that each supermartingale family is right-USCE. Hence, since J and J' are supermartingale families, they are right-USCE and so are $J - \zeta$ and $J' + \xi$ because ξ and $-\zeta$ are right-USCE.

Recall now that (see (24) in the Appendix), $(J'(\sigma), \sigma \in T_{[\theta, \sigma^\lambda]})$ is a martingale family. Hence, since J is a supermartingale family, it follows that $Y = J - J'$ is a supermartingale family. We thus have

$$Y(\theta) \geq E[Y(\sigma^\lambda \wedge \tau) | \mathcal{F}_\theta] \geq E[Y(\tau)\mathbb{1}_{\tau \leq \sigma^\lambda} + Y(\sigma^\lambda)\mathbb{1}_{\sigma^\lambda < \tau} | \mathcal{F}_\theta] \text{ a.s.} \quad (15)$$

Recall now that $Y \geq \xi$. Moreover, thanks to inequality (23) in the Appendix, we have the following inequality $\lambda J'(\sigma^\lambda) \leq J(\sigma^\lambda) - \zeta(\sigma^\lambda)$ a.s., which can be written

$$Y(\sigma^\lambda) \geq \zeta(\sigma^\lambda) - (1 - \lambda)J'(\sigma^\lambda) \text{ a.s.}$$

This with inequality (15) leads to

$$Y(\theta) \geq E[\xi(\tau)\mathbb{1}_{\tau \leq \sigma^\lambda} + \zeta(\sigma^\lambda)\mathbb{1}_{\sigma^\lambda < \tau} | \mathcal{F}_\theta] - (1 - \lambda)E[J'(\sigma^\lambda)\mathbb{1}_{\sigma^\lambda \leq \tau} | \mathcal{F}_\theta] \text{ a.s.}$$

The supermartingale property of J' yields that

$$E[J'(\sigma^\lambda)\mathbb{1}_{\sigma^\lambda \leq \tau} | \mathcal{F}_\theta] \leq E[J'(\sigma^\lambda) | \mathcal{F}_\theta] \leq J'(\theta) \text{ a.s.}$$

This with the previous inequality and the definition of $I_\theta(\tau, \sigma^\lambda)$ leads to

$$Y(\theta) \geq I_\theta(\tau, \sigma^\lambda) - (1 - \lambda)J'(\theta) \text{ a.s.}$$

By the same arguments, one can show the following inequality:

$$Y(\theta) \leq I_\theta(\tau^\lambda, \sigma) + (1 - \lambda)J(\theta) \text{ a.s.}$$

The proof of Lemma 3.3 is thus complete. \square

Remark 3.4 — Let \bar{J} and \bar{J}' be two nonnegative supermartingale families satisfying integrability conditions (1) and such that $\bar{J} = \mathcal{R}(\bar{J}' + \xi)$ and $\bar{J}' = \mathcal{R}(\bar{J} - \zeta)$. The same proof shows that the above property still holds for \bar{J} and \bar{J}' .

More precisely, if $\bar{J}(0) < +\infty$ and if ξ and $-\zeta$ are right-USCE, then $Y(\theta) = \bar{J}(\theta) - \bar{J}'(\theta)$ a.s. and equalities (13) hold.

3.1 The right-continuous in expectation case

Let us now introduce the following definition.

Definition 3.5 — An admissible family $(\phi(\theta), \theta \in \mathcal{T})$ is said to be *right-continuous in expectation along stopping times (right-CE)* if for all $\theta \in \mathcal{T}$ and for all sequences of stopping times (θ_n) such that $\theta^n \downarrow \theta$, $E[\phi(\theta)] = \lim_{n \rightarrow \infty} E[\phi(\theta_n)]$.

We first show that the limit of a non decreasing sequence of right-CE supermartingale families is also an right-CE supermartingale family. More precisely, the following property holds.

Lemma 3.6 — For each $n \in \mathbb{N}$, let $\phi_n := (\phi_n(\theta), \theta \in \mathcal{T})$ be a right-CE nonnegative supermartingale family. Suppose that the sequence $(\phi_n)_{n \in \mathbb{N}}$ is non decreasing that is, for each $n \in \mathbb{N}$ and each $\theta \in \mathcal{T}$, $\phi_n(\theta) \leq \phi_{n+1}(\theta)$ a.s.

The family ϕ defined for each $\theta \in \mathcal{T}$ by $\phi(\theta) := \limsup_{n \rightarrow \infty} \phi_n(\theta)$ is then an right-CE supermartingale family.

Proof — By Lemma 2.5, we already know that family ϕ is a supermartingale family. It remains to show that it is right-CE. Let $\theta \in \mathcal{T}$ and let (θ_p) be a sequence of stopping times such that $\theta^p \downarrow \theta$. By the monotone convergence theorem and the right-CE property of ϕ_n , we have

$$\begin{aligned} \lim_{p \rightarrow \infty} \uparrow E[\phi(\theta_p)] &= \lim_{p \rightarrow \infty} \uparrow E[\lim_{n \rightarrow \infty} \uparrow \phi_n(\theta_p)] \\ &= \lim_{p \rightarrow \infty} \uparrow \lim_{n \rightarrow \infty} \uparrow E[\phi_n(\theta_p)] = \lim_{n \rightarrow \infty} \uparrow \lim_{p \rightarrow \infty} \uparrow E[\phi_n(\theta_p)] \\ &= \lim_{n \rightarrow \infty} \uparrow E[\lim_{p \rightarrow \infty} \uparrow \phi_n(\theta_p)] = E[\phi(\theta)]. \end{aligned}$$

The proof is thus complete. \square

Remark 3.7 — We stress on the simplicity of the proof. Note also that using the above lemma together with the well-known result of aggregation of right-CE supermartingales (see Th. 3.13 in Karatzas and Shreve (1994)), one can easily derive the analogous result for processes (see Th. 18 ch. VI in Dellacherie Meyer (1980) as well as their quite long proof, comparing to ours).

Proposition 3.8 — Suppose that $J(0) < +\infty$ and that the families ξ and $-\zeta$ are right-CE. Then, the families J and J' are right-CE. Also, $Y (= J - J')$ is the common value function of the saddle point problem and is right-CE.

Proof — Recall that by a classical result of optimal stopping theory (see Lemma 2.13 in El Karoui (1981)), the value function associated with an right-CE reward family is right-CE. This ensures that by induction, for each n , J_n and J'_n are right-CE. Since $J = \lim_{n \rightarrow \infty} \uparrow J_n$ and $J' = \lim_{n \rightarrow \infty} \uparrow J'_n$, by Lemma 3.6, J and J' are right-CE.

Moreover, by Theorem 3.2, since ξ and ζ are RCE and hence right-USCE, it follows that $Y = \underline{V} = \bar{V}$ that is, Y is the common value function of the saddle point problem. Also, since $Y = J - J'$, it is right-CE. \square

4 Existence of saddle points

Under additional assumptions, we provide an existence result.

First, by similar arguments as those used in the proof of Proposition 2.7, we have

Proposition 4.1 — Let ξ and ζ be two admissible families satisfying integrability conditions (1). The following conditions are equivalent

- The family J is of class \mathcal{D} .
- The family J' is of class \mathcal{D} .
- The strong Mokobodski condition holds that is, there exist two nonnegative supermartingale families H and H' of class \mathcal{D} such that

$$\xi \leq H - H' \leq \zeta.$$

We now introduce the following definition.

Definition 4.2 — An admissible family $(\phi(\theta), \theta \in \mathcal{T})$, satisfying $E[\text{ess sup}_{\theta \in \mathcal{T}} (\phi(\theta))^-] < +\infty$ and of class \mathcal{D} , is said to be *strongly left-upper semicontinuous along stopping times in expectation* (strong left-USCE) if for all $\theta \in \mathcal{T}$, for all $F \in \mathcal{F}_{\theta-}$ and for all non decreasing sequences of stopping times $(\theta_n)_{n \in \mathbb{N}}$ such that $\theta^n \uparrow \theta$,

$$\limsup_{n \rightarrow \infty} E[\phi(\theta_n) \mathbb{1}_F] \leq E[\phi(\theta) \mathbb{1}_F]. \quad (16)$$

Remark 4.3 — Note that in this definition, no condition is required at a totally inaccessible stopping time (such as, for example, a jump of a Poisson process). Indeed, suppose that θ is a totally inaccessible stopping time. Then, if $(\theta_n)_{n \in \mathbb{N}}$ is a non decreasing sequence of stopping times converging to θ , it is necessarily a.s. constant equal to θ from a certain rank. Using the integrability conditions and Fatou's lemma, we get

$$\limsup_{n \rightarrow \infty} E[\phi(\theta_n) \mathbb{1}_F] \leq E[\limsup_{n \rightarrow \infty} \phi(\theta_n) \mathbb{1}_F] = E[\phi(\theta) \mathbb{1}_F].$$

Hence, inequality (16) is always satisfied for any $F \in \mathcal{F}_{\theta-}$.

In the particular case of an optional process (ϕ_t) , the strong left-USCE property of the family $(\phi_\theta, \theta \in \mathcal{T})$ is thus weaker than the usual left-upper semicontinuity property of the process (ϕ_t) .

We provide the following regularity result.

Theorem 4.4 — Let ξ and ζ be two admissible families satisfying integrability conditions (1) and such that the families J and J' are of class \mathcal{D} . Suppose that for each predictable stopping time $\tau \in \mathcal{T}$ with $\tau > 0$ a.s.

$$\{\xi(\tau) = \zeta(\tau)\} = \emptyset \quad \text{a.s.} \quad (17)$$

If ξ and $-\zeta$ are right-USCE and strong left-USCE, the families J and J' are then left-USCE.

This theorem together with Proposition 2.9 and the existence result for optimal stopping time problem (see (26) in the Appendix or [9]) provides the following general existence result.

Corollary 4.5 — (**Existence result**) Suppose that the assumptions of the above theorem hold. For each $\theta \in \mathcal{T}$, the stopping time

$$\tau_*(\theta) := \text{ess inf} \{ \tau \in \mathcal{T}_\theta, J(\tau) = J'(\tau) + \xi(\tau) \text{ a.s.} \}.$$

is an optimal stopping time for $J(\theta)$ and

$$\sigma_*(\theta) := \text{ess inf} \{ \sigma \in \mathcal{T}_\theta, J'(\sigma) = J(\sigma) - \zeta(\sigma) \text{ a.s.} \}.$$

is an optimal stopping time for $J'(\theta)$. Moreover, the pair $(\tau_*(\theta), \sigma_*(\theta))$ is a θ -saddle point for the criterion I_θ and

$$Y(\theta) = \underline{V}(\theta) = \bar{V}(\theta) = I_\theta(\tau_*(\theta), \sigma_*(\theta)) \quad \text{a.s.}$$

Remark 4.6 — Let us consider the particular case where the families ξ and ζ are given by predictable processes (ξ_t) and (ζ_t) . Then, by classical results on processes (see Dellacherie and Meyer (1975)), condition (17) is equivalent to the fact that $P(\xi_t = \zeta_t, \forall t) = 0$. Of course, this equivalence does not hold if (ξ_t) and (ζ_t) are only supposed to be optional. Indeed, in general, in order to ensure that $P(\xi_t = \zeta_t, \forall t) = 0$, condition (17) must hold for any optional stopping time τ .

Note also that our assumptions are weaker than those made in Alario-Nazaret, Lepeltier and Marchal (1982) (see section 2 p30). Moreover, their proof of the left-upper semicontinuity property of the processes J and J' (see Lemma 4-2 p30 in their paper) requires highly sophisticated results of the General Theory of Processes as, among others, the so called Mertens decomposition of supermartingales and the existence of a left-upper semicontinuous envelope \bar{X} for a given optional process X . On the contrary, the proof given below is only based on classical properties of Probability Theory.

Before showing Theorem 4.4, we provide the following lemma.

Lemma 4.7 — *Suppose that for each predictable stopping time $\tau \in \mathcal{T}$ with $\tau > 0$ a.s., equality (17) is satisfied. Then, for each $\theta \in \mathcal{T}$ and for each non decreasing sequence of stopping times $(\theta_n)_{n \in \mathbb{N}}$ such that $\theta^n \uparrow \theta$, we have*

$$\{\xi(\theta) = \zeta(\theta)\} \cap \{\theta_n < \theta, \text{ for all } n\} = \emptyset \quad \text{a.s.}$$

Proof of Lemma 4.7 — Let us introduce the set $A := \{\theta_n < \theta, \text{ for all } n\}$.

Let us first show that θ coincides on A with a predictable stopping time. Let $\tau := \lim_{n \rightarrow \infty} \theta_n$ and let

$$\tau_n := (\theta_n \mathbb{1}_{\{\theta_n < S\}} + T \mathbb{1}_{\{\theta_n \geq S\}}) \wedge (T - \frac{1}{n}),$$

for each n . The sequence (τ_n) announces its limit τ everywhere on Ω . In particular, τ is predictable. Also, $\tau = \theta$ a.s. on A . Hence, we get

$$\{\xi(\theta) = \zeta(\theta)\} \cap A = \{\xi(\tau) = \zeta(\tau)\} \cap A \subset \{\xi(\tau) = \zeta(\tau)\} = \emptyset \quad \text{a.s.},$$

which provides the desired result. \square

Proof of Theorem 4.4 — Let θ be a given stopping time and let $(\theta_n)_{n \in \mathbb{N}}$ be a non decreasing sequence of stopping times such that $\theta^n \uparrow \theta$. Let us show that $\limsup_{n \rightarrow +\infty} E[J(\theta_n)] \leq E[J(\theta)]$.

Let us $A := \{\theta_n < \theta, \text{ for all } n\}$. Since for almost every $\omega \in A^c$, the sequence $(\theta_n(\omega))$ is constant from a certain rank, it follows that the sequence $(J(\theta_n)(\omega))$ is also constant from a certain rank. Indeed, note first that $A^c := \cup_p \cap_{l \geq p} \{\theta_l = \theta\}$. Let $p \in \mathbb{N}$. By the admissibility property of Z , for each $n \geq p$, $J(\theta_n) = J(\theta)$ a.s. on $\cap_{l \geq p} \{\theta_l = \theta\}$. Hence, $\lim_{n \rightarrow \infty} J(\theta_n) = J(\theta)$ a.s. on $\cap_{l \geq p} \{\theta_l = \theta\}$ and this holds for each p . Hence, $\lim_{n \rightarrow \infty} J(\theta_n) = J(\theta)$ a.s. on A^c .

Since the family J is of class \mathcal{D} , we have $\lim_{n \rightarrow +\infty} E[J(\theta_n) \mathbb{1}_{A^c}] = E[J(\theta) \mathbb{1}_{A^c}]$. It is thus sufficient to show that

$$\limsup_{n \rightarrow +\infty} E[J(\theta_n) \mathbb{1}_A] \leq E[J(\theta) \mathbb{1}_A].$$

Since (θ_n) announces θ on A , by the convergence theorem for nonnegative discrete supermartingales, the sequence $(J(\theta_n))_{n \in \mathbb{N}}$ converges a.s. to a nonnegative random variable denoted by $J(\theta^-)$. Also, $J(\theta^-) \mathbb{1}_A$ is \mathcal{F}_{θ^-} -measurable and, if $(\theta'_n)_{n \in \mathbb{N}}$ is a non decreasing sequence of stopping times such that $\theta'_n \uparrow \theta$, then $\lim_{n \rightarrow \infty} J(\theta'_n) = \lim_{n \rightarrow \infty} J(\theta_n) = J(\theta^-)$ a.s. on $A \cap A'$, where $A' := \{\theta'_n < \theta, \text{ for all } n\}$, as precised in the Appendix (see also Theorem 4.5 in Kobylanski and Quenez (2011) for more details).

Similarly, there exists a nonnegative random variable which we denote $J'(\theta^-)$ such that $\lim_{n \rightarrow \infty} J'(\theta_n) = J'(\theta^-)$ a.s. This random $J'(\theta^-)$ satisfies similar properties as $J(\theta^-)$.

Since the family J is of class \mathcal{D} , we have $\lim_{n \rightarrow +\infty} E[J(\theta_n) \mathbb{1}_A] = E[J(\theta^-) \mathbb{1}_A]$. The problem thus reduces to prove that

$$E[(J(\theta) - J(\theta^-)) \mathbb{1}_A] \geq 0. \tag{18}$$

Let $B := \{E[J(\theta) | \mathcal{F}_{\theta^-}] < J(\theta^-)\}$. We have

$$\begin{aligned} E[(J(\theta) - J(\theta^-)) \mathbb{1}_A] &= E[(E[J(\theta) | \mathcal{F}_{\theta^-}] - J(\theta^-)) \mathbb{1}_A] \\ &= E[(E[J(\theta) | \mathcal{F}_{\theta^-}] - J(\theta^-)) \mathbb{1}_{A \cap B}] \\ &= E[(J(\theta) - J(\theta^-)) \mathbb{1}_{A \cap B}] \end{aligned}$$

Recall that for each p and for each $\lambda \in [0, 1[$, $\theta^\lambda(\theta_p) < \theta$ a.s. on $B \cap A$ (see Lemma A.3). Moreover, by Lemma A.4,

$$\begin{aligned} E[J(\theta^-) \mathbb{1}_{A \cap B}] &= E[J'(\theta^-) \mathbb{1}_{A \cap B}] + \sup_{\lambda \in [0, 1[} \limsup_{p \rightarrow \infty} E[\xi(\tau^\lambda(\theta_p)) \mathbb{1}_{A \cap B}] \\ &\leq E[J'(\theta^-) \mathbb{1}_{A \cap B}] + E[\xi(\theta) \mathbb{1}_{A \cap B}], \end{aligned}$$

where the last inequality follows from the inequality

$$\limsup_{p \rightarrow \infty} E[\xi(\tau^\lambda(\theta_p)) \mathbb{1}_{A \cap B}] = \limsup_{p \rightarrow \infty} E[\xi(\tau^\lambda(\theta_p) \wedge \theta) \mathbb{1}_{A \cap B}] \leq E[\xi(\theta) \mathbb{1}_{A \cap B}],$$

due to the strong left-USCE property of ξ (see (16)). It follows that

$$E[(J(\theta) - J(\theta^-)) \mathbb{1}_{A \cap B}] \geq E[(J(\theta) - J'(\theta^-) - \xi(\theta)) \mathbb{1}_{A \cap B}].$$

Let $B' := \{E[J'(\theta) | \mathcal{F}_{\theta^-}] < J'(\theta^-)\}$. Suppose now that we have shown that

$$A \cap B \cap B' = \emptyset \quad \text{a.s.} \quad (19)$$

This yields that $A \cap B \subset (B')^c$ a.s., which implies that

$$E[J'(\theta) | \mathcal{F}_{\theta^-}] = J'(\theta^-) \quad \text{a.s. on } A \cap B.$$

Hence,

$$\begin{aligned} E[(J(\theta) - J(\theta^-)) \mathbb{1}_{A \cap B}] &\geq E[(J(\theta) - J'(\theta^-) - \xi(\theta)) \mathbb{1}_{A \cap B}] \\ &\geq E[(J(\theta) - J'(\theta) - \xi(\theta)) \mathbb{1}_{A \cap B}] \geq 0 \quad \text{a.s.}, \end{aligned}$$

since $J(\theta) \geq J'(\theta) + \xi(\theta)$ a.s. Hence, J is left-USCE. By similar arguments, we have that J' is also left-USCE.

It remains to prove (19). Let $C := A \cap B \cap B'$. By Lemma A.4 and since ξ is strong left-USCE, we have

$$\begin{aligned} E[J(\theta^-) \mathbb{1}_C] &= E[J'(\theta^-) \mathbb{1}_C] + \sup_{\lambda \in [0, 1[} \limsup_{p \rightarrow \infty} E[\xi(\tau^\lambda(\theta_p)) \mathbb{1}_C] \\ &\leq E[J'(\theta^-) \mathbb{1}_C] + E[\xi(\theta) \mathbb{1}_C]. \end{aligned}$$

Similarly, since $-\zeta$ is strong left-USCE,,

$$\begin{aligned} E[J'(\theta^-) \mathbb{1}_C] &= E[J(\theta^-) \mathbb{1}_C] - \inf_{\lambda \in [0, 1[} \liminf_{p \rightarrow \infty} E[\zeta(\sigma^\lambda(\theta_p)) \mathbb{1}_C] \\ &\leq E[J(\theta^-) \mathbb{1}_C] - E[\zeta(\theta) \mathbb{1}_C]. \end{aligned}$$

By adding the two above inequalities, we get

$$0 \leq E[\xi(\theta) \mathbb{1}_C] - E[\zeta(\theta) \mathbb{1}_C],$$

which, with the inequality $\zeta(\theta) \geq \xi(\theta)$ a.s., leads to $\xi(\theta) = \zeta(\theta)$ a.s. on C . Since, by Lemma 4.7, $\{\xi(\theta) = \zeta(\theta)\} \cap A = \emptyset$ a.s., it follows that $P(C) = 0$. The proof of Theorem 4.4 is thus complete. \square

A Appendix

A.1 Some results on optimal stopping in the framework of families of random variables

Let $(\phi(\theta), \theta \in \mathcal{T})$ be an admissible family called *reward* satisfying the following integrability condition:

$$E[\text{ess sup}_{\theta \in \mathcal{T}} \phi(\theta)^-] < +\infty. \quad (20)$$

For $S \in \mathcal{T}_0$, the *value function at time S* is given by

$$v(S) = \text{ess sup}_{\theta \in \mathcal{T}_S} E[\phi(\theta) | \mathcal{F}_S]. \quad (21)$$

This optimal stopping problem clearly reduces to the case of a nonnegative reward, which has been studied by Kobylanski and Quenez (2011) in the framework of families of random variables.

More precisely, define $X(\theta) := E[\text{ess sup}_{\tau \in \mathcal{T}_\theta} \phi(\tau)^- | \mathcal{F}_\theta]$ and $\bar{\phi}(\theta) := \phi(\theta) + X(\theta)$. This new reward $\bar{\phi}$ is nonnegative and the associated new value function \bar{v} satisfies $\bar{v}(\theta) = v(\theta) + X(\theta)$ a.s. By translation, all the properties satisfied by the new value function \bar{v} are thus satisfied by the value function v .

First, v is a supermartingale family and is characterized as the Snell envelope family associated with the reward family ϕ that is, the smallest supermartingale family which is greater (a.s.) than ϕ .

In the sequel, we denote by \mathcal{R} the *Snell* envelope operator defined on the set of admissible families satisfying condition (20) and valued in the set of supermartingale families. For each admissible family ϕ satisfying (20), $\mathcal{R}(\phi)$ is the corresponding value function v or equivalently the associated *Snell* envelope family.

Note also that if the family ϕ is of class \mathcal{D} (uniformly integrable), then $\mathcal{R}(\phi)$ is of class \mathcal{D} .

Let $S \in \mathcal{T}$ and let $\theta_* \in \mathcal{T}_S$ be such that $E[\phi(\theta_*)] < \infty$. The stopping time θ_* is said to be S -optimal if $v(S) = E[\phi(\theta_*) | \mathcal{F}_S]$ a.s. The so called *optimality criterion* ensures that θ_* is S -optimal if and only if

$$v(\theta_*) = \phi(\theta_*) \quad \text{a.s.} \quad \text{and} \quad (v(\theta), \theta \in \mathcal{T}_{[S, \theta_*]}) \quad \text{is a martingale family.} \quad (22)$$

Recall the following results (proven in Kobylanski and Quenez (2011) section 2.1.). Suppose the reward $(\phi(\theta), \theta \in \mathcal{T})$ is right-USCE and $v(0) < \infty$. Let $S \in \mathcal{T}$ and $\lambda \in]0, 1[$. The stopping time $\theta^\lambda(S)$ defined by

$$\theta^\lambda(S) := \text{ess inf} \{ \theta \in \mathcal{T}_S, \lambda v(\theta) \leq \phi(\theta) \text{ a.s.} \}$$

satisfies

$$\lambda v(\theta^\lambda(S)) \leq \phi(\theta^\lambda(S)) \quad \text{a.s.} \quad (23)$$

and

$$v(S) = E[v(\theta^\lambda(S)) | \mathcal{F}_S] \quad \text{a.s.} \quad (24)$$

Note that this equality is equivalent to the martingale property of the family $(v(\theta), \theta \in \mathcal{T}_{[S, \theta^\lambda(S)]})$. From (23) and (24), it follows that $\theta^\lambda(S)$ is $(1 - \lambda)$ -optimal for $v(S)$ that is,

$$\lambda v(S) \leq E[\phi(\theta^\lambda(S)) | \mathcal{F}_S] \quad \text{a.s.} \quad (25)$$

Moreover, under the additional assumption of left-USCE property of the reward ϕ , the $(1 - \lambda)$ -optimal stopping times $\theta^\lambda(S)$ tend to an optimal stopping time for $v(S)$ as $\lambda \uparrow 1$. More precisely (see Kobylanski and Quenez (2011) section 2.2.1), for each $S \in \mathcal{T}$, the stopping time $\theta_*(S)$ defined by

$$\theta_*(S) := \text{ess inf} \{ \theta \in T_S, v(\theta) = \phi(\theta) \text{ a.s.} \} \quad (26)$$

is the minimal optimal stopping time for $v(S)$. Moreover, $\theta_*(S) = \lim_{\lambda \uparrow 1} \theta^\lambda(S)$ a.s.

At last, recall some regularity results which are used in the proof of Theorem 4.4. Note first that, without any assumption on the reward family, the value function is right-USCE because it is a supermartingale.

For S in \mathcal{T} . Suppose that $S \in T_{0+}$ that is, $S > 0$ a.s. Note that, since \mathcal{F}_0 is the trivial σ -algebra augmented with the P -null sets, this condition is equivalent to $P(S > 0) > 0$. Recall that a non decreasing sequence of stopping times $(S_n)_{n \in \mathbb{N}}$ such that for each n , $S_n \leq S$ a.s., is said to *announce* S on $A \subset \Omega$ if

$$S_n \uparrow S \text{ a.s. on } A \text{ and } S_n < S \text{ a.s. on } A.$$

We have the following property, which is used in the proof of Theorem 4.4.

Proposition A.1 — *Let $(u(\theta), \theta \in \mathcal{T})$ be a supermartingale family satisfying $E[\text{ess sup}_{\theta \in \mathcal{T}} u(\theta)^-] < +\infty$.*

Let S be a stopping time in \mathcal{T}_{0+} . Suppose that S is accessible on a subset A of Ω .

There exists an \mathcal{F}_{S-} -measurable random variable $u(S^-)$, unique on A (up to the equality a.s.), such that, for any non decreasing sequence $(S_n)_{n \in \mathbb{N}}$ announcing S on A , one has

$$u(S^-) = \lim_{n \rightarrow \infty} u(S_n) \quad \text{a.s. on } A.$$

If $u(0) < +\infty$, then $u(S^-)\mathbb{1}_A$ is integrable.

Remark A.2 — Recall that, by definition, the set $A(S)$ of accessibility of S is the union of the sets on which S is accessible. By a result of Dellacherie and Meyer (Chap IV.80), there exists a sequence of sets $(A_k)_{k \in \mathbb{N}}$ in \mathcal{F}_{S-} such that for each k , S is accessible on A_k and $A(S) = \cup_k A_k$ a.s. It follows that $u(S^-)$ is well defined on $A(S)$ and depends only on S . Actually, this property is not used in this paper.

Proof — The proof is based on classical results of probability theory. Without loss of generality, we can suppose that u is nonnegative. Let $(S_n)_{n \in \mathbb{N}}$ be a non decreasing sequence announcing S on A . It is clear that $(u(S_n))_{n \in \mathbb{N}}$ is a discrete nonnegative supermartingale relatively to the filtration $(\mathcal{F}_{S_n})_{n \in \mathbb{N}}$. By the well-known convergence theorem for discrete supermartingales, there exists a random variable Z such that $(u(S_n))_{n \in \mathbb{N}}$ converges a.s. to Z . If $u(0) < +\infty$, then $Z\mathbb{1}_A$ is integrable. We then set $u(S^-) := Z$.

It remains to show that this limit $u(S^-)$, restricted to A , does not depend on the sequence (S_n) . For this proof, one is referred to Kobylanski and Quenez (2011) Theorem 4.5. \square

A.2 Two useful lemmas

We now provide two lemmas which are used in the proof of Theorem 4.4. The proofs are based on classical results of probability theory.

Suppose $(\phi(\theta), \theta \in \mathcal{T})$ is a right-USCE admissible family satisfying $E[\text{ess sup}_{\theta \in \mathcal{T}} \phi(\theta)^-] < +\infty$ and $v(0) < \infty$.

Let $\theta \in \mathcal{T}$ and let $(\theta_p)_{p \in \mathbb{N}}$ in \mathcal{T} such that $\theta_p \uparrow \theta$.

Suppose that the event $A := \{\theta_n < \theta, \text{ for all } n\}$ is non empty and set $B := \{E[v(\theta) | \mathcal{F}_{\theta-}] < v(\theta^-)\}$, where $v = \mathcal{R}(\phi)$. We provide the following result.

Lemma A.3 — *For each p and for each $\lambda \in [0, 1[$, $\theta^\lambda(\theta_p) < \theta$ a.s. on $B \cap A$.*

Proof — For this, it is sufficient to show that for each p , $B \cap A \cap \{\theta^\lambda(\theta_p) \geq \theta\} = \emptyset$ a.s. Note first that $\{\theta^\lambda(\theta_p) \geq \theta\} = \cap_q \{\theta^\lambda(\theta_p) \geq \theta_q\}$. Hence, $\{\theta^\lambda(\theta_p) \geq \theta\} \in \mathcal{F}_{\theta-} \cap \bigvee_n \mathcal{F}_{\theta_n}$. Also, for each $q \geq p$, $E[v(\theta^\lambda(\theta_p)) | \mathcal{F}_{\theta_q}] = v(\theta_q)$ a.s. on $\{\theta^\lambda(\theta_p) \geq \theta_q\}$ and hence on $\{\theta^\lambda(\theta_p) \geq \theta\}$, because $(v(\tau), \tau \in T_{[\theta_p, \theta^\lambda(\theta_p)]})$ is a martingale family. Hence, by letting q tend to ∞ ,

$$E[v(\theta^\lambda(\theta_p)) | \bigvee_n \mathcal{F}_{\theta_n}] = v(\theta^-) \quad \text{a.s. on } \{\theta^\lambda(\theta_p) \geq \theta\} \cap A.$$

Now, we have

$$E[v(\theta^\lambda(\theta_p)) | \bigvee_n \mathcal{F}_{\theta_n}] = E[v(\theta^\lambda(\theta_p)) | \mathcal{F}_{\theta-}] \quad \text{a.s. on } A.$$

It follows that

$$E[v(\theta^\lambda(\theta_p)) | \mathcal{F}_{\theta-}] = v(\theta^-) \quad \text{a.s. on } \{\theta^\lambda(\theta_p) \geq \theta\} \cap A. \quad (27)$$

By (24), we have $E[v(\theta^\lambda(\theta_p)) | \mathcal{F}_\theta] = v(\theta)$ a.s. on $\{\theta^\lambda(\theta_p) \geq \theta\}$. Hence, by taking the conditional expectation with respect to $\mathcal{F}_{\theta-}$, we derive that

$$E[v(\theta^\lambda(\theta_p)) | \mathcal{F}_{\theta-}] = E[v(\theta) | \mathcal{F}_{\theta-}] < v(\theta^-) \quad \text{a.s. on } B \cap A \cap \{\theta^\lambda(\theta_p) \geq \theta\},$$

which, with equality (27), yields that $B \cap A \cap \{\theta^\lambda(\theta_p) \geq \theta\} = \emptyset$ a.s. \square

Recall now that, by (23), for each $\lambda \in [0, 1[$ and for each $p \in \mathbb{N}$, we have

$$\lambda v(\theta^\lambda(\theta_p)) \leq \phi(\theta^\lambda(\theta_p)) \quad \text{a.s.} \quad (28)$$

From this property, we derive the following property.

Lemma A.4 — *If ϕ is of class \mathcal{D} , we have*

$$E[v(\theta^-) \mathbb{1}_{A \cap B}] = \sup_{\lambda \in [0, 1[} \limsup_{p \rightarrow \infty} E[\phi(\theta^\lambda(\theta_p)) \mathbb{1}_{A \cap B}].$$

Also, for each $D \in \mathcal{F}_{\theta-}$, this equality holds with $A \cap B$ replaced by $A \cap B \cap D$.

Proof — From inequality (28), it follows that for each $\lambda \in [0, 1[$ and each $p \in \mathbb{N}$,

$$\lambda E[v(\theta^\lambda(\theta_p)) \mathbb{1}_{A \cap B}] \leq E[\phi(\theta^\lambda(\theta_p)) \mathbb{1}_{A \cap B}].$$

By letting p tend to $+\infty$, we derive that

$$\lambda \limsup_{p \rightarrow \infty} E[v(\theta^\lambda(\theta_p)) \mathbb{1}_{A \cap B}] \leq \limsup_{p \rightarrow \infty} E[\phi(\theta^\lambda(\theta_p)) \mathbb{1}_{A \cap B}].$$

Since $(\theta^\lambda(\theta_p) \wedge \theta)$ announces θ on $A \cap B$, by Proposition A.1, it follows that $\lim_{n \rightarrow \infty} v(\theta^\lambda(\theta_p) \wedge \theta) = v(\theta^-)$ a.s. on $A \cap B$. Using Fatou's lemma, we get

$$E[v(\theta^-) \mathbb{1}_{A \cap B}] \leq \liminf_{p \rightarrow \infty} E[v(\theta^\lambda(\theta_p)) \mathbb{1}_{A \cap B}] \leq \limsup_{p \rightarrow \infty} E[v(\theta^\lambda(\theta_p)) \mathbb{1}_{A \cap B}].$$

The two above inequalities yield that

$$\lambda E[v(\theta^-)\mathbb{1}_{A \cap B}] \leq \limsup_{p \rightarrow \infty} E[\phi(\theta^\lambda(\theta_p))\mathbb{1}_{A \cap B}],$$

and this holds for each $\lambda \in [0, 1[$. Hence, by taking the supremum over $\lambda \in]0, 1[$, we get

$$E[v(\theta^-)\mathbb{1}_{A \cap B}] \leq \sup_{\lambda \in]0, 1[} \limsup_{p \rightarrow \infty} E[\phi(\theta^\lambda(\theta_p))\mathbb{1}_{A \cap B}].$$

The same arguments show that for each $D \in \mathcal{F}_{\theta^-}$, this inequality holds with $A \cap B$ replaced by $A \cap B \cap D$.

At last, by using the inequality $v \geq \phi$ and the fact that v is of class \mathcal{D} , we derive that the inequality is an equality. \square

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